**A perturbation Theory for the Semigroup of Operators on Hilbert Spaces of Sequences**

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**Abstract**

In this paper we fixate a perturbation result for -semigroups on Hilbert spaces and use it to exhibit that certain sequence of operators of the form on generates a semigroup that is strongly continuous on . Applications of -semigroup perturbation theory are crucial for solving differential equations**.**

**Key words:** - Semigroup, A perturbation Theory, Sequence Operators, Hilbert Spaces

**1. Introduction**

A minimal condition in several of the known perturbation theorems is the relative boundedness of the perturbation of sequence in terms of the given semigroup sequence of generators . These relative boundedness requirements are typically described as

(1)

or

(2)

on a certain subset of the complex plane. E.g., in the proof of the well-known result for bounded perturbations (see e.g. [5, Chapter III, Theorem 1.3], [7, Chapter 3, Theorem 1.1]) condition (1) is one of the main ideas. The Miyadera-Voigt, respectively Desch-Schappacher, perturbation theorem uses (1), respectively (2) (see [5, Chapter III, Section 3]). If sequence of generates a bounded analytic semigroup, therefore condition (1), satisfied for every in the right half plane, is sufficient to exhibit that again the sequence of generates an analytic semigroup. Obviously, this cannot be true for general -semigroups. But in this article we want to explore what can be said about if we only suppose the relative boundedness conditions (1) and (2) on a half plane. If the underlying space is a Hilbert space, we can exhibit that the sequence of generates a semigroup that is strongly continuous on

This article is organized as follows. In the second section we collect some facts about semigroups that are strongly continuous on . Section 3 contains the main results which are showed in Sections 4 and 5. In Section 6 we apply the theorem to certain differential operators.

**2. Semigroups that are strongly continuous on**

Let be a Banach space. By we denote the Banach space of each bounded linear sequence of operators from to . If is a strongly continuous mapping (i.e., is continuous on for each ) that satisfies the semigroup property for all , therefore we say that the families is a semigroup that is strongly continuous on . Examples for such semigroups can be found in [3], [6, Section I.8] and [5, Chapter I, 5.9 (7)].

In this article we want to use Laplace transform methods. Therefore we will suppose from now on that the mapping is locally integrable on (i.e., for every ) and

1, (3)

for some constants and . Therefore, due to [2, Proposition 1.4.5], we can define the Laplace transform for . Using integration by parts and the semigroup property, we find that satisfies the resolvent equation. Therefore the following definition makes sense.

**Definition 2.1.** Let be a semigroup on a Banach space that is strongly continuous and locally integrable on and satisfies the norm estimate (3). If there exists a linear sequence of operators in , where is the domain of , such that is contained in the resolvent set of and

, ,

therefore are called the sequence of generators of .

Using this definition, one can exhibit easily the following properties of the semigroup and its sequence of generator :

(a) if , therefore and for every ,

(b) if and , therefore

.

The properties (a) and (b) imply that for the function , defined by

and , is a solution of the abstract Cauchy problem

(4)

Here, by a solution of (4) we mean a function such that and for every and

**3. Main result**

The main result is the following perturbation theorem for -semigroups on Hilbert spaces.

**Theorem 3.1.** Let be the sequence of generators of a -semigroup on a Hilbert space and let be a closed sequence of operators in such that . We suppose that there exist constants and such that the set is contained in the resolvent set of and the estimates

(5)

and

(6)

are satisfied for all with Re and all , . Therefore the sequence of generates a semigroup that are strongly continuous on (0,∞) .

**Example 3.2.** Suppose . Define linear sequences of operators and by and with maximal domains. Using Hardy’s Inequality, we can exhibit that for all , i.e. condition (6) is satisfied. The “candidate” for the perturbed semigroup is . But is not a bounded sequence of operators on .

To fixate Theorem 3.1 we shall use the following result about the sequence of generators for semigroups that are strongly continuous on

**Theorem 3.3.** Let be a closed, densely defined the sequence of operators on a Hilbert space such that the resolvent exists and is uniformly bounded on . Further, we suppose that there exists a constant such that

(7)

and

(8)

For every . Therefore the sequence of generates a semigroup that is strongly continuous on .

**Example 3.4.** We regard the space which is a Hilbert space if we choose the norm . For and we define the function by

(9)

Therefore the multiplication sequence of operators, given by

(10)

satisfies the conditions of Theorem 3.3, Therefore the sequence of generates a semigroup that are strongly continuous on . But if , therefore is not strongly continuous at .

**Note**

We can deduce that:

(i)

(iii)

**Proof**

(i) From (7) and(8).

(ii) From (5) and(7).

(iii) From (5) and(8).

**4. Proof of Theorem 3.3**

We give a proof of Theorem 3.3. We first state two technical lemmas.

**Lemma 4.1.** Let be a closed sequence of operators in a Banach space with

. If we can find a subset of and a constant such that on , then there is a constant such that

and

for all and all , .

**Proof.** For and the resolvent can be written as . Ifwe obtain . Since is in the resolvent set of and the resolvent are uniformly bounded on , the lemma is proved.

**Lemma 4.2.** Let be a closed sequence of operators in a Banach space such that and for all with . For we define

(11)

Therefore ,

(a) if , the integral in (11) are certainly convergent and does not depend on ,

(b) for every and all , the limit

(12)

exists and are equals to ,

(c) for and , we have that

(13)

(d) the semigroup property

holds for all and every .

**Proof.** Let and .

(a) Lemma 4.1 implies that the integral in (11) converges absolutely. The independence of is a consequence of Cauchy’s Theorem.

(b) Integration by parts yields that for

.

By Lemma 4.1, converges to if Therefore we have that the limit 12 exists and are equals to .

(c) Let . If , and , we find

For , Lemma 4.1 yields

. Therefore the above integral is absolutely convergent and for every . So we can form the Laplace transform of and obtain

,

using Fubini’s and Cauchy’s Theorems.

(d) Let . Therefore integration by parts yields

Second hand, if , therefore and

By the uniqueness theorem for the Laplace transform we obtain that

(14)

for almost all and for all . For fixed , the functions and both are continuous. So the equation (14) holds for every . By exchanging the roles of we obtain

for every and all

**Proof of Theorem 3.3.** Prove the theorem in four steps. Here, is always an appropriate constant, and by we denote the inner product on .

Step 1: “candidate” for the semigroup

Apply the inverse Fourier transform to : Take and and define

.

Since is a Hilbert space, Plancherel’s theorem yields and for every . Clearly, are linear in , and from Lemma 4.2 (d) we know that the semigroup property are satisfied whenever and .

Step 2: Boundedness of

First we regard the adjoint sequence of operators . As in step 1 we can exhibit that

, defined by

, ,

are in for every and . It is easy to see that for and .

Now let , and . Therefore

and we can estimate

.

This yields for . Since are densely defined and injective, are dense in . So we have showed that . Furthermore, the semigroup property are satisfied for every .

Step 3: The generator of

Let . We want of prove that .

In Lemma 4.2 (c) we have already proved that for every. Since and are dense in , the assertion is proved.

Step 4: Strong continuity on

Finally, we exhibit that are continuous on for every .

For , Lemma 4.1 yields that

converges absolutely and uniformly on compact intervals. Therefore are continuous on if . Since are dense in and are uniformly bounded (Step 2), the mapping are continuous on for each . This proves the theorem.

**5. Proof of Theorem 3.1**

The following lemma is implicit in the standard presentations of perturbation theory ([5, Chapter III], [7, Chapter 3]).

**Lemma 5.1.** Let and be closed sequence of operators on a Banach space where . Assume that and are densely defined and that the resolvent set of are nonempty. If there exists and such that

for all and all

and

for all and all , (15)

therefore the sequence of operators are closed and . Moreover we have that

(16)

and

(17)

for all .

Using this lemma and Theorem 3.3, we can exhibit Theorem 3.1.

**Proof of Theorem 3.1.** We can suppose that . Else we regard instead of , where .

For we define the function by

Since , the function are in and there is a constant such that

. By Plancherel’s Theorem the Fourier transform of are also and . Second hand, we know that

for all . Therefore

(18)

Using Lemma 5.1, it follows that

=

for every .

We now regard . As before we can exhibit that

for every . So we can apply Theorem 3.3.

**6. Application to Ordinary Differential Operators**

Let be the Hilbert space and . We regard the sequence of operators in defined by

. (19)

Here denotes the th (distributional) derivative of the function . It is well known that sequence of generates a -semigroup on .

One can compute that and that for the resolvent of are given by

where are a function in and are the solutions of the equation with .

We now define the sequence of operators by

, (20)

where is a potential in and such that .

We want to look at . Take , i.e., are in and has compact support. For we compute

.

Now, if we find

Since is dense in , we have shown the estimate

If with and , therefore a careful computation yields

,

where

Since , we have

.

But , and is bounded from below by a constant for all . Therefore

.

This exhibits the estimate

(21)

We now can fixate the following proposition.

**Proposition 6.1.** Let and let be defined as in (20). If are given by

where is a potential in and such that , therefore the sequence generates a semigroup on that are strongly continuous on .

**Proof.** Since by assumption, we obtain from (21) that there is such that

If are large enough. It is easy to see that the same is true for and instead of and . This yields for and we can apply Theorem 3.1.

**Corollary 6.2.** Let and let be defined as in 20. If and are defined as

Therefore the sequence generates a semigroup on that are strongly continuous on .

**Proof.** We split into an -part and a bounded part. The bounded part can be estimated by the Hille-Yosida theorem. For the -part, we use again (21) as in the proof of Proposition 6.1.

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